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Optimal Dividend Strategy in Dual Risk Model with Tax

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ABSTRACT
This paper investigates optimal dividend strategies in a dual risk model where dividend payments are subject to taxation. We consider a surplus process driven by a compound Poisson process with negative drift and study the optimization problem under bounded dividend rates. A tax mechanism is incorporated where dividends below a threshold are tax-free, while those above face proportional taxation. We establish verification theorems and derive explicit solutions for exponentially distributed income sizes using two-level threshold strategies. Numerical examples demonstrate the effectiveness of our approach.

KEYWORDS
Dual risk model; Optimal dividends; Tax policy; Hamilton-Jacobi-Bellman equation; Threshold strategy

1 Introduction

The dual risk model, where the surplus process has negative drift and positive jumps, has gained significant attention in actuarial science and finance. Unlike the classical Cramér-Lundberg model, the dual model captures scenarios where an entity (such as a pharmaceutical company or startup) faces continuous expenses while receiving sporadic income.

This paper extends the dual risk model by incorporating dividend payments subject to a realistic tax structure. We assume dividends below threshold β_1 are tax-free, while those exceeding β_1 but below β_2 face proportional taxation at rate ($1-k$).

Our main contributions include: (1) establishing verification theorems for the Hamilton-JacobiBellman (HJB) equation, (2) deriving explicit solutions using two-level threshold strategies when incomes are exponentially distributed, and (3) providing numerical illustrations.

2 Model formulation

2.1 The dual risk process

Consider a surplus process without dividend control:

$$dU(t) = -cdt + dS(t),\ U(0) = x \geq 0 \tag{1}$$

where $c>0$ represents the constant expenditure rate, and $S(t)=\sum_{i=1}^{N_t} X_i$ is a compound Poisson process with intensity $\lambda>0$. The income sizes $\{X_i\}$ are i.i.d. non-negative random variables with density $p(y)$, independent of the Poisson process $\{N_t\}$.

2.2 Dividend control and taxation

Let $l_\pi(t)\in[0,\beta_2]$ denote the dividend rate at time t under strategy π. The controlled surplus process becomes:

$$dU_\pi(t) = -\left(c + l_\pi(t)\right)dt + dS(t) \tag{2}$$

The after-tax dividend utility function is:

$$g(l) = \begin{cases} l, & \text{if } l \in \left(0, \beta_1\right] \\ \beta_1 + k\left(l - \beta_1\right), & \text{if } l \in \left(\beta_1, \beta_2\right] \end{cases} \tag{3}$$

where $0<k<1$ represents the tax efficiency parameter.

2.3 Optimization Problem

Define the ruin time $\tau_\pi = \inf\left\{t \geq 0 : U_\pi(t) \leq 0\right\}$. The value function is:

$$V(x) = \sup_{\pi \in \Pi} J_\pi(x) \tag{4}$$

where

$$J_\pi(x)=E_x\left[\int_0^{\tau_\pi} e^{-\delta t} g\left(l_\pi(t)\right) dt\right] \tag{5}$$

and $\delta>0$ is the discount rate.

3 Main results

3.1 Hamilton-jacobi-bellman equation

If $V(x)$ is differentiable, it satisfies the HJB equation:

$$\sup_{l\in[0,\beta_2]} A_l V(x)=0 \tag{6}$$

with $V(0)=0$, where

$$A_l m(x)=-(c+l)m'(x)-(\lambda+\delta)m(x)+\lambda\int_0^\infty m(x+y)p(y)dy+g(l) \tag{7}$$

3.2 Verification theorems

Theorem 1*Suppose* $m(x)$:$[0,\infty)\to[0,\infty)$ *is increasing, concave, bounded, and continuously differentiable, and satisfies* $\max_{0\leqslant l\leqslant\beta_2} A_l m(x)\leqslant 0$ *for all* $x\ge 0$. *Then* $m(x)\ge V(x)$ *for all* $x\ge 0$.

Theorem 2*For any* $x>0, V(x)\le\dfrac{\beta_1+k\left(\beta_2-\beta_1\right)}{\delta}$.

4 Exponential income case

Assume incomes follow exponential distribution: $F_Y(y)=1-e^{-\beta y}, y\ge 0$.

The optimal control structure is:

$$l^*(x)=\begin{cases}0, & \text{if } m'(x)>1\\ \in\left[0,\beta_1\right], & \text{if } m'(x)=1\\ \beta_1, & \text{if } k<m'(x)<1\\ \in\left[\beta_1,\beta_2\right], & \text{if } m'(x)=k\\ \beta_2, & \text{if } m'(x)<k\end{cases} \tag{8}$$

This leads to three cases based on threshold values.

4.1 Case 1: threshold pair (0,0)

When $m'(0)\le k$, the optimal strategy pays maximum dividends for all $x\ge 0$:

$$m(x)=\frac{\beta_1+k\left(\beta_2-\beta_1\right)}{\delta}\left(1-e^{h_2 x}\right) \tag{9}$$

where $h_2<0$ is the negative root of:

$$\left(c+\beta_2\right)y^2+\left[(\lambda+\delta)-\beta\left(c+\beta_2\right)\right]y-\delta\beta=0 \tag{10}$$

4.2 Case 2: threshold pair $(0,w_2)$

When $k<m'(0)\le 1$, there exists $w_2>0$ such that:

$$m(x)=\begin{cases}B_1 e^{d_1 x}+B_2 e^{d_2 x}+\dfrac{\beta_1}{\delta}, & 0\le x\le w_2\\ B_3 e^{h_2 x}+\dfrac{\beta_1+k\left(\beta_2-\beta_1\right)}{\delta}, & x>w_2\end{cases} \tag{11}$$

The characteristic roots $d_1>0$ and $d_2<0$ satisfy:

$$d_{1,2}=\frac{\beta\left(c+\beta_1\right)-(\lambda+\delta)\pm\sqrt{\left[\beta\left(c+\beta_1\right)-(\lambda+\delta)\right]^2+4\delta\beta\left(c+\beta_1\right)}}{2\left(c+\beta_1\right)} \tag{12}$$

The coefficients are determined by continuity and smoothness conditions:

$$B_1 = k\frac{1-\eta d_2}{d_1-d_2}e^{-d_1 w_2},\ B_2 = k\frac{\eta d_1-1}{d_1-d_2}e^{-d_2 w_2} \tag{13}$$

$$B_3 = \frac{k}{h_2}e^{-h_2 w_2} \tag{14}$$

where $\eta = \frac{1}{h_2}+\frac{\beta_2-\beta_1}{\delta}$, and the threshold w_2 satisfies:

$$k\frac{1-\eta d_2}{d_1-d_2}e^{-d_1 w_2}+k\frac{\eta d_1-1}{d_1-d_2}e^{-d_2 w_2}+\frac{\beta_1}{\delta}=0 \tag{15}$$

4.3 Case 3: threshold pair (w_1,w_2)

When $m'(0)>1$, the solution involves two thresholds $0<w_1<w_2$:

$$m(x)=\begin{cases} B_4\left(e^{c_1 x}-e^{c_2 x}\right), & 0\le x\le w_1 \\ B_5 e^{d_1 x}+B_6 e^{d_2 x}+\dfrac{\beta_1}{\delta}, & w_1\le x\le w_2 \\ B_7 e^{h_2 x}+\dfrac{\beta_1+k\left(\beta_2-\beta_1\right)}{\delta}, & x>w_2 \end{cases} \tag{16}$$

The characteristic roots $c_1>0$ and $c_2<0$ are:

$$c_{1,2}=\frac{[\beta c-(\lambda+\delta)]\pm\sqrt{[c\beta-(\lambda+\delta)]^2+4\delta\beta c}}{2c} \tag{17}$$

Let $u_3=\frac{1-\eta d_2}{d_1-d_2}>0$ and $u_4=-\frac{1-\eta d_1}{d_1-d_2}<0$. The coefficients are:

$$B_5 = ku_3 e^{-d_1 w_2},\ B_6 = ku_4 e^{-d_2 w_2} \tag{18}$$

$$B_7=\frac{k}{h_2}e^{-h_2 w_2} \tag{19}$$

The thresholds satisfy $w_1-w_2=w^*$ where $w^*<0$ is the unique solution to:

$$ku_3 d_1 e^{d_1 w^*}+ku_4 d_2 e^{d_2 w^*}=1 \tag{20}$$

Let $v_1=ku_3 e^{d_1 w^*}+ku_4 e^{d_2 w^*}$. Then:

$$w_1=\frac{1}{c_1-c_2}\log\frac{\left(v_1+\frac{\beta_1}{\delta}\right)c_2-1}{\left(v_1+\frac{\beta_1}{\delta}\right)c_1-1} \tag{21}$$

$$B_4=\frac{1}{c_1\left[\frac{\left(v_1+\frac{\beta_1}{\delta}\right)c_2-1}{\left(v_1+\frac{\beta_1}{\delta}\right)c_1-1}\right]^{\frac{c_1}{c_1-c_2}}-c_2\left[\frac{\left(v_1+\frac{\beta_1}{\delta}\right)c_2-1}{\left(v_1+\frac{\beta_1}{\delta}\right)c_1-1}\right]^{\frac{c_2}{c_1-c_2}}} \tag{22}$$

5 Numerical example

Consider parameters: $k=0.2,\beta_2=0.8,\beta_1=0.2,\delta=0.05,\lambda=2,c=1.2,\beta=0.5$.

Case 1: $\left(w_1,w_2\right)=(0,0)$

$$m(x)=6.4\left(1-e^{-0.5478x}\right) \tag{23}$$

Case 2: $w_2=3.6801$

With $B_1=2.0556, B_2=-6.0556, B_3=-2.7412$:

$$m(x)=\begin{cases} 2.0556e^{0.0182x}-6.0556e^{-0.9825x}+4, & 0\le x\le 3.6801 \\ -2.7412e^{-0.5478x}+6.4, & x>3.6801 \end{cases} \tag{24}$$

Case 3: $\left(w_1,w_2\right)=(1.6747,3.4995)$

With $B_4=5.7170, B_5=2.0624, B_6=-5.0712, B_7=-2.4830$:

$$m(x)=\begin{cases} 5.7170\left(e^{0.0170x}-e^{-1.2253x}\right), & 0\le x\le 1.6747 \\ 2.0624e^{0.0182x}-5.0712e^{-0.9825x}+4, & 1.6747<x\le 3.4995 \\ -2.4830e^{-0.5478x}+6.4, & x>3.4995 \end{cases} \tag{25}$$

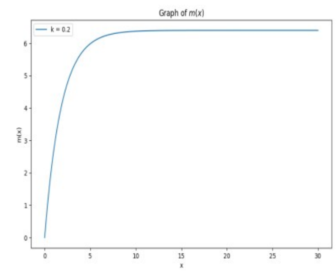

Figure 1 Maximum dividend strategy

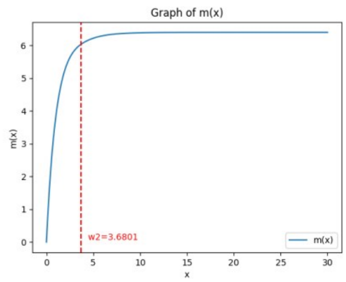

Figure 2 Two-threshold strategy

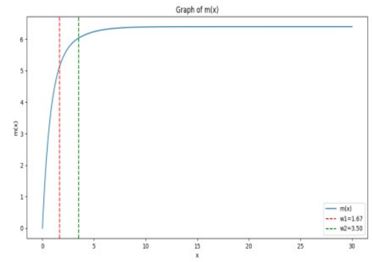

Figure 3 Three-region strategy

All cases yield increasing, bounded, continuously differentiable concave solutions verifying the optimality conditions.

The figures above illustrate the value function $m(x)$ for different parameter configurations, allowing for direct comparison between the three strategies. The left panel shows the maximum dividend rate strategy where dividends are paid at the highest allowable rate for all surplus levels. The middle panel demonstrates the two-threshold strategy with distinct regions for tax-free and taxed dividend payments. The right panel presents the three-region strategy with no-dividend, tax-free dividend, and taxed dividend regions, representing the most complex optimal dividend policy.

6 Conclusion

We have developed a comprehensive analysis of optimal dividend strategies in dual risk models with tax considerations. Our main contributions include establishing verification theorems and providing explicit solutions for exponentially distributed incomes using two-level threshold strategies. The explicit coefficient expressions demonstrate the practical computability of the optimal strategies.

The tax structure significantly influences the optimal policy, creating distinct regimes based on the relationship between marginal utility and tax rates. The numerical examples with complete coefficient specifications demonstrate the practical applicability of our theoretical results.

Future research could extend this framework to incorporate more complex tax structures, stochastic interest rates, or regime-switching dynamics.

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